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WKB approximation in deformed space with minimal length

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Abstract
The WKB approximation for deformed space with minimal length is considered. The Bohr–Sommerfeld quantization rule is obtained. A new interesting feature in the presence of deformation is that the WKB approximation is valid for intermediate quantum numbers and can be invalid for small as well as very large quantum numbers. The correctness of the rule is verified by comparing obtained results with exact expressions for corresponding spectra.

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1. Introduction

Quantum mechanics with modification of the usual canonical commutation relations has been investigated intensively lately. Such works are motivated by several independent lines of investigation in string theory and quantum gravity, which suggest the existence of a finite lower bound to the possible resolution of length ΔX [1–3].

A lot of attention was paid to the following deformed commutation relation [4–7]:

$$[X, P] = i\hbar(1 + \beta P^2) \tag{1}$$

and it was shown that it implies the existence of minimal resolution length $\Delta X = \sqrt{\langle(\Delta X)^2\rangle} \geqslant \hbar\sqrt{\beta}$ [4], i.e. there is no possibility of measuring coordinate X with an accuracy larger than ΔX. If someone puts $\beta = 0$, the usual Heisenberg algebra can be obtained.

The use of the deformed commutation relations (1) brings new difficulties in solving the quantum problems. As far as we know there are only a few problems for which spectra have been found exactly. They are the one-dimensional oscillator [4], the D-dimensional isotropic harmonic oscillator [8], the three-dimensional relativistic Dirac oscillator [9], and 1D Coulomb potential [10]. Note that in the one-dimensional case, the harmonic oscillator problem has been solved exactly [11, 12] for more general deformation leading to nonzero uncertainties in both position and momentum.

Difficulties of obtaining exact solutions of quantum problems lead to the development of perturbation techniques [5, 13–15] and numerical calculus [15] in the presence of deformation. In our recent work [10], we derived the exact expression for the spectrum of 1D Coulomb potential and we also obtained the same result with the help of the Bohr–Sommerfeld quantization rule. The use of the rule was intuitional; in this paper we derive this rule rigidly and analyse its applicability.

The present paper is organized as follows. In the second section, WKB approximation is extended for deformed commutation relation, Bohr–Sommerfeld quantization rule is obtained and its applicability is discussed. In the third section, several 1D examples are analysed and obtained spectra are compared to the exact results. In the fourth section, we show that the quantization rule can be applied to 3D problems with radial symmetry. The paper ends with concluding remarks.

2. WKB approximation

The 1D Schrödinger equation in deformed space reads

$$\left[\frac{P^2}{2m} + U(X)\right]\psi = E\psi, \tag{2}$$

where the first term in the brackets describes kinetic energy of the system and the second term describes the potential. In the deformed space, coordinate and momentum operators satisfy the following commutation relation:

$$[X, P] = \mathrm{i}\hbar f(P), \tag{3}$$

where $f(P)$ is an arbitrary function of P, in general $f(P) \neq 1$. We require that $f(P)$ to be an even function. This provides invariance of relation (3) with respect to the reflection $X \to -X, P \to -P$. Here we consider general form of deformation, particular form of deformation (1) will be analysed in more details at the end of the section.

In order to study the semiclassical approximation, we use the so-called quasi-coordinate representation

$$X = x, \qquad P = P(p), \qquad p = -\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}x}. \tag{4}$$

From definitions (3) and (4), we obtain

$$\frac{\mathrm{d}P(p)}{\mathrm{d}p} = f(P) \tag{5}$$

and $P(p)$ is an odd function.

Let us express wavefunction in the following form:

$$\psi(x) = \exp\left[\frac{\mathrm{i}}{\hbar}S(x)\right] \tag{6}$$

then in linear approximation over $\hbar$

$$P^2\psi(x) = \left[P^2(S'(x)) - \frac{\mathrm{i}\hbar}{2}[P^2(S'(x))]''S''(x) + \cdots\right]\psi(x), \tag{7}$$

here prime denotes derivative with respect to the argument of the function. Expanding $S(x)$ in power series over $\hbar$

$$S(x) = S_0(x) + \frac{\hbar}{\mathrm{i}}S_1(x) + \cdots \tag{8}$$

we obtain the following set of equations for S:

$$\frac{P^2(S_0'(x))}{2m} + U(x) = E, \tag{9}$$

$$\frac{[P^2(S_0'(x))]'}{2m}\frac{\hbar}{\mathrm{i}}S_1'(x) - \frac{\mathrm{i}\hbar}{4m}[P^2(S_0'(x))]''S_0''(x) = 0. \tag{10}$$

Solutions of these equations read

$$S_0(x) = \int^x p(\pm\sqrt{2m(E-U(x))})\,\mathrm{d}x = \pm\int^x p(P)\,\mathrm{d}x, \tag{11}$$

$$S_1(x) = -\frac{1}{2}\ln|[P^2(S_0'(x))]'| = -\frac{1}{2}\ln|2P(x)f(P)|, \tag{12}$$

where $P = \sqrt{2m(E-U(x))}$ is a function of x, $p(P)$ is an inverse function to $P(p)$; it is taken into account that $p(P)$ is also the odd function. Here we omit constants of integration since they are taken into account in the final expression for the wavefunction:

$$\psi(x) = \frac{1}{\sqrt{|Pf(P)|}}\left(C_1\exp\left[\frac{\mathrm{i}}{\hbar}\int^x p\,\mathrm{d}x\right] + C_2\exp\left[-\frac{\mathrm{i}}{\hbar}\int^x p\,\mathrm{d}x\right]\right). \tag{13}$$

To obtain an expression of the Bohr–Sommerfeld quantization rule, we have to analyse the behaviour of wavefunction (13) at the infinities and consider matching conditions near the turning points. For bound states this analysis leads to the following Bohr–Sommerfeld quantization condition:

$$\int_{x_1}^{x_2} p\,\mathrm{d}x = \pi\hbar(n+\delta), \qquad n = 0, 1, 2, \ldots \tag{14}$$

where x_1 and x_2 are turning points satisfying equation $U(x) = E$, δ depends on boundary conditions and properties of $P(x)$. If potential $U(x)$ is a smooth function and if $f(0) \neq 0$ then $\delta = 1/2$.

Note that new small operators x and p satisfy canonical commutation relation; therefore condition (14) is expected and was used in our previous work [10] as an evident one. In that paper the following recipe to find a spectrum with the help of Bohr–Sommerfeld quantization rule was applied: we rewrite the problem in the small operators x and p which satisfy $[x, p] = \mathrm{i}\hbar$, then from equation $H(x, p) = E$ we find out $p = p(x, E)$ and the use Bohr–Sommerfeld quantization rule (14).

The quantization rule (14) can be rewritten in more convenient form using the following transformations:

$$\oint p\,\mathrm{d}x = -\oint x\,\mathrm{d}p = -\oint x\,\mathrm{d}P\frac{\mathrm{d}p}{\mathrm{d}P}. \tag{15}$$

Then taking into account expressions (4) and (5), we obtain equivalent form of the Bohr–Sommerfeld quantization rule

$$-\oint\frac{X\,\mathrm{d}P}{f(P)} = 2\pi\hbar(n+\delta). \tag{16}$$

This rule does not demand knowing of representation of initial operators X and P in terms of canonical operators x and p and can be applied to an eigenvalue problem at once.

The WKB approximation is valid if the second term of expansion (7) is much less than the first term. Namely, it is valid if

$$P^2 \gg \frac{\hbar}{2}|[P^2(S'(x))]''S''(x)| \approx \frac{\hbar}{2}\left|\frac{d^2P^2}{\mathrm{d}p^2}\frac{\mathrm{d}p}{\mathrm{d}x}\right|, \tag{17}$$

here we substitute $S'(x)$ with $p = S_0'(x)$. This substitution is correct in linear approximation over $\hbar$. Then, using the fact that $\frac{d^2 P^2}{dp^2} = \frac{d}{dp}(2Pf(P))$, we obtain

$$P^2 \gg \hbar \left| \frac{d}{dx} Pf(P) \right|. \tag{18}$$

In the undeformed case, it is considered that the WKB approximation is valid for large quantum number n. In the case of deformation, condition (18) can be violated for large values of quantum numbers. We analyse this violation in more details for special case of deformation

$$f(P) = (1 + \beta P^2). \tag{19}$$

This case corresponds to

$$P(p) = \frac{1}{\sqrt{\beta}} \tan \sqrt{\beta} p, \qquad p(P) = \frac{1}{\sqrt{\beta}} \arctan \sqrt{\beta} P. \tag{20}$$

Such a deformation (19), (20) for smooth potential energy $U(x)$ gives $\delta = 1/2$.

Condition (18) reads

$$P^2 \gg \hbar(1 + 3\beta P^2) \left| \frac{dP}{dx} \right|. \tag{21}$$

For small momentum ($\beta P^2 \ll 1$), we obtain usual condition for validity of the WKB approximation [17]:

$$\hbar \left| \frac{d(1/P)}{dx} \right| \ll 1. \tag{22}$$

For large P, we obtain that the following inequality must hold:

$$3\hbar\beta \left| \frac{dP}{dx} \right| \ll 1. \tag{23}$$

Let us use rough approximation

$$\left| \frac{dP}{dx} \right| \approx \frac{P}{a} = \frac{2\pi\hbar}{\lambda a}, \tag{24}$$

where a is a characteristic size of the system being about $x_2 - x_1$, λ is a wavelength corresponding to momentum P. It allows with the use of formulae (22), (23) to estimate ranges in which WKB approximation is valid:

$$a \gg \lambda \gg \frac{\Delta X^2}{a}, \tag{25}$$

where $\Delta X = \hbar\sqrt{\beta}$ is a minimal resolution length. It is interesting to note that if $a \approx \Delta X$ then WKB approximation is not valid for any momentum value. This result is not an unexpected one because a characteristic size of the system must be larger than minimal resolution length, otherwise all mathematics and physics become meaningless.

3. 1D examples

3.1. Harmonic oscillator

The Hamiltonian of the system is

$$H = P^2 + X^2, \tag{26}$$

here and below we put $m = 1/2$, $\omega = 2$ and $\hbar = 1$ for the sake of simplicity.

From $H(P, X) = E$, we obtain

$$X = \sqrt{E - P^2} \tag{27}$$

and Bohr–Sommerfeld quantization condition (16) gives

$$2\int_{-\sqrt{E}}^{\sqrt{E}} \frac{\sqrt{E - P^2}}{1 + \beta P^2}\, \mathrm{d}P = \frac{2\pi}{\beta}(\sqrt{1 + \beta E} - 1) = 2\pi(n + 1/2). \tag{28}$$

From the last equation we obtain

$$E_n = (2n + 1) + \beta\left(n^2 + n + \tfrac{1}{4}\right). \tag{29}$$

The exact result obtained by Kempf and collaborators [4] is

$$E_n = (2n + 1)\left(\frac{\beta}{2} + \sqrt{1 + \frac{\beta^2}{4}}\right) + \beta n^2 \approx (2n + 1) + \beta\left(n^2 + n + \frac{1}{2}\right) + O(\beta^2). \tag{30}$$

As one can see, results presented by formulae (29) and (30) asymptotically coincide for large n.

The inequality (25) for harmonic oscillator reads

$$\sqrt{n} \gg \frac{1}{\sqrt{n}} \gg \frac{\beta}{\sqrt{n}}. \tag{31}$$

So if $\beta \ll 1$, it simplifies to usual condition of WKB approximation applicability

$$n \gg 1.$$

3.2. *Anharmonic oscillator*

The Hamiltonian reads

$$H = P^2 + \gamma^N X^N, \tag{32}$$

where N is an even integer.

Then from $H(P, X) = E$, we obtain

$$X = \frac{1}{\gamma}(E - P^2)^{1/N} \tag{33}$$

and

$$2\int_{-\sqrt{E}}^{\sqrt{E}} \frac{X\, \mathrm{d}P}{1 + \beta P^2} = 2\pi(n + 1/2). \tag{34}$$

We calculate the last integral in linear approximation over β and obtain that

$$E_n = E_n^0\left(1 + \frac{2\beta}{(1 + 2/N)(3 + 2/N)}E_n^0\right), \tag{35}$$

where E_n^0 denotes energy levels obtained using Bohr–Sommerfeld quantization rule for $\beta = 0$ and they read

$$E_n^0 = \left[\pi \frac{\Gamma(3/2 + 1/N)}{\Gamma(1/2)\Gamma(1 + 1/N)}\gamma(n + 1/2)\right]^{\frac{2N}{2+N}}. \tag{36}$$

If $N = 2$ and $\gamma = 1$, we reproduce result (29) obtained for 1D harmonic oscillator.

In limit $N \to \infty$, we obtain system which is equivalent to infinitely high potential well. For this case (35) gives

$$E_n = \left(\frac{\pi\gamma\left(n + \frac{1}{2}\right)}{2}\right)^2 + \frac{2}{3}\beta\left(\frac{\pi\gamma\left(n + \frac{1}{2}\right)}{2}\right)^4, \tag{37}$$

where $2/\gamma$ stands for the well width. Previously the problem has been considered in [7] for general form of deformation function $f(P)$ and in linear approximation over β their approach for $f(P) = 1 + \beta P^2$ gives

$$E_n = \left(\frac{\pi n \gamma}{2}\right)^2 + \frac{2}{3}\beta\left(\frac{\pi n \gamma}{2}\right)^4. \tag{38}$$

The difference between formulae (37) and (38) appears due to the limiting procedure ($\delta = 1/2$ for finite N, $\delta = 0$ for $N = \infty$). Direct consideration of potential well in WKB approximation gives the same result as in (38).

Expression (25) gives the following condition:

$$1 \ll n \ll \frac{1}{\gamma^2 \beta}$$

of WKB approximation applicability for infinitely high potential well ($N \to \infty$). One can see that the approximation is not valid for small n (as in undeformed case), but it becomes invalid for very large n.

3.3. $-1/X^2$ potential

Let us consider the following Hamiltonian:

$$H = P^2 - \frac{\gamma}{X^2}, \qquad \gamma > 0. \tag{39}$$

Only for negative energies there exist bound states. From equation $H(X, P) = E$, we find

$$X = \frac{\sqrt{\gamma}}{\sqrt{P^2 - E}}. \tag{40}$$

Then rule (16) can be rewritten as

$$4\int_0^\infty \frac{\sqrt{\gamma}\,\mathrm{d}P}{\sqrt{P^2 - E}(1 + \beta P^2)} = \frac{\sqrt{\gamma}}{\sqrt{1 + E\beta}} \ln\left(\frac{2 + 2\sqrt{1 + E\beta} + E\beta}{2 - 2\sqrt{1 + E\beta} + E\beta}\right) = 2\pi(n + \delta), \tag{41}$$

where δ depends on boundary conditions. Equation (41) can be solved for small β and it gives

$$E_n = -\frac{4}{\beta}\,\mathrm{e}^{-\pi(n+\delta)/\sqrt{\gamma}}. \tag{42}$$

For $\beta \to 0$ one can see that $E_n \to -\infty$. It corresponds to the fact that for undeformed case, there does not exist any bound state for this potential. So, deformation of the space leads to the existence of bound states for $-1/X^2$ potential. For a singular potential $V(X)$, we can estimate $P \approx \sqrt{2m|V(X)|}$ at the vicinity of the singularity point. It is easy to show that inequality (18) does not hold for potentials $-1/X^2$ and $-1/X$ either in deformed or in undeformed spaces at the vicinity of the origin. So, formally the WKB approximation cannot be applied to $-1/X^2$ potential. On the other hand, the Bohr–Sommerfeld quantization rule gives exact result for $-1/X$ potential in deformed space [10]. In the undeformed case, the WKB approximation also can be applied to singular potentials (see for an instance [16]). Thus, we may expect that obtained spectrum (42) is quite accurate too.

4. 3D examples

In the second section, we prove the Bohr–Sommerfeld quantization rule (14), (16) for 1D case, then in the next section we illustrated the rule with the examples. The examples presented in this section demonstrate that we can use Bohr–Sommerfeld rule in 3D space and obtain satisfactory results.

Deformed commutation relation usually is generalized to the following form in the 3D case [13]:

$$[X_i, P_j] = \mathrm{i}(1 + \beta P^2)\delta_{ij} + \mathrm{i}\beta' P_i P_j. \tag{43}$$

There exists a simple momentum representation for coordinate and momentum operators

$$X_i = (1 + \beta p^2)x_i + \beta' p_i \sum_{j=1}^{3} p_j x_j, \qquad P_i = p_i. \tag{44}$$

In a semiclassical approach, coordinate and momentum operators are substituted with corresponding variables and

$$X^2 = [1 + (\beta + \beta')p^2]^2 x_p^2 + (1 + \beta p^2)^2 \frac{L^2}{p^2}, \tag{45}$$

here we use spherical system of coordinates (p, θ, ϕ); $x_p = \frac{(\vec{x}\cdot\vec{p})}{p}$ denotes 'radial part' of coordinate, L^2 is an angular part. The 3D problem in semiclassical approach can be reduced to a 1D one if one substitutes L^2 with $(l + 1/2)^2$ [17].

4.1. Hydrogen atom

The classical Hamiltonian reads

$$H(p, x) = P^2 - \frac{\gamma}{X} = p^2 - \frac{\gamma}{\sqrt{[1 + (\beta + \beta')p^2]^2 x_p^2 + (1 + \beta p^2)^2 \left(\frac{l+1/2}{p}\right)^2}}. \tag{46}$$

The energy values of bound states of hydrogen atom are negative. Then from the equation $H(p, x) = E$, we obtain

$$x_p = \frac{1}{1 + (\beta + \beta')p^2} \sqrt{\frac{\gamma^2}{(p^2 - E)^2} - (1 + \beta p^2)^2 \left(\frac{l + 1/2}{p}\right)^2} \tag{47}$$

and Bohr–Sommerfeld quantization condition reads

$$2 \int_{p_{\min}}^{p_{\max}} x_p \, \mathrm{d}p = 2\pi \left(n + \frac{1}{2}\right). \tag{48}$$

The integral (48) is very cumbersome, so we expand it in powers of β and β'. In linear approximation, it reads

$$\int_{p_{\min}}^{p_{\max}} x_p \, \mathrm{d}p \approx \int_{p_{\min}}^{p_{\max}} \sqrt{\ldots} \, \mathrm{d}p - (\beta + \beta') \int_{p_{\min}}^{p_{\max}} \sqrt{\ldots}\, p^2 \, \mathrm{d}p - \beta \left(l + \frac{1}{2}\right)^2 \int_{p_{\min}}^{p_{\max}} \frac{\mathrm{d}p}{\sqrt{\ldots}}, \tag{49}$$

where

$$\sqrt{\ldots} = \sqrt{\frac{\gamma^2}{(p^2 - E)^2} - \left(\frac{l + 1/2}{p}\right)^2}.$$

The integration of (49) gives

$$-\pi \left(l + \frac{1}{2}\right) + \frac{\gamma\pi}{2\sqrt{-E}} - \pi(\beta + \beta')\gamma \left(\frac{\gamma}{4(l + 1/2)} - \frac{\sqrt{-E}}{2}\right) - \pi\beta \frac{\gamma^2}{4} \frac{1}{l + 1/2}. \tag{50}$$

Then solution of equation (48) in linear approximation gives

$$E_{n,l} \approx -\frac{\gamma^2}{4n^2} + \frac{\gamma^4}{8n^3} \left(\beta \left[\frac{2}{l + 1/2} - \frac{1}{n}\right] + \beta' \left[\frac{1}{l + 1/2} - \frac{1}{n}\right]\right). \tag{51}$$

We compare this result with expression for the correction obtained by Benczik and collaborators [15] with the help of perturbative theory. Their expression contains one additional term

$$\frac{\gamma^4}{16n^3}\frac{2\beta - \beta'}{l(l+1)(l+1/2)}.$$

For large l, the term is small in comparison with the rest terms. So, for 3D hydrogen atom Bohr–Sommerfeld quantization rule provides satisfactory accuracy.

4.2. Harmonic oscillator

The Hamiltonian of the system is

$$H = P^2 + X^2 = p^2 + [1 + (\beta + \beta')p^2]^2 x_p^2 + (1 + \beta p^2)^2 \frac{(l+1/2)^2}{p^2}. \tag{52}$$

From equation $H(p, x) = E$, we obtain

$$x_p = \frac{1}{1 + (\beta + \beta')p^2}\sqrt{E - p^2 - (1 + \beta p^2)^2\left(\frac{l+1/2}{p}\right)^2}. \tag{53}$$

Corresponding contour integral can be calculated exactly but it is very cumbersome. On the other hand, as we see Bohr–Sommerfeld quantization rule gives correct result only in linear approximation over β, β'. Therefore, we calculate it in the linear approximation

$$\frac{\pi}{2}E - \frac{\pi}{2}(\beta - \beta')(l+1/2)^2 - \pi(l+1/2) - \frac{\pi}{8}(\beta + \beta')E^2 = 2\pi(n_p + 1/2), \tag{54}$$

from which we obtain

$$E_n = 2n + 3 + (\beta + \beta')(n + 3/2)^2 + (\beta - \beta')(l + 1/2)^2, \tag{55}$$

where $n = 2n_p + l$. The spectrum of 3D harmonic oscillator was calculated exactly in [8]. The difference of their exact expression and our approximate one is

$$2\beta - \frac{\beta'}{2}. \tag{56}$$

For large n and l, we see the method is again good.

5. Concluding remarks

In this paper, we derived the Bohr–Sommerfeld quantization rule and considered its applicability to the 1D case. A new interesting feature appearing in the presence of deformation is that WKB approximation becomes valid for intermediate quantum numbers, but it can become invalid for small (as in the undeformed case) as well as for very large quantum numbers. This feature is illustrated with example 3.2, an infinitely high potential well.

To verify the method, we compared results obtained with the help of the Bohr–Sommerfeld quantization rule with exact spectra expressions for the harmonic oscillator and the infinitely high potential well (examples 3.1, 3.2) and showed that the results obtained are asymptotically exact for large n. The consideration of $-1/X^2$ potential indicates that there may exist bound states for this potential in deformed space, although for the undeformed case bound states do not exist (example 3.3). It was shown that the method could be applied to 3D problems with radial symmetry with satisfactory accuracy (see examples 4.1, 4.2).

As a result it seems that the Bohr–Sommerfeld quantization rule can be applied to consideration of a wide variety 1D problems as well as to 3D problems in deformed space.

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References

[1] Gross D J and Mende P F 1988 *Nucl. Phys.* B **303** 407
[2] Maggiore M 1993 *Phys. Lett.* B **304** 65
[3] Witten E 1996 *Phys. Today* **49** 24
[4] Kempf A, Mangano G and Mann R B 1995 *Phys. Rev.* D **52** 1108
[5] Kempf A and Mangano G 1997 *Phys. Rev.* D **55** 7909
[6] Kempf A 2000 *Phys. Rev. Lett.* **85** 2873
[7] Detournay S, Gabriel C and Spindel Ph 2002 *Phys. Rev.* D **66** 125004
[8] Chang L N, Minic D, Okamura N and Takeuchi T 2002 *Phys. Rev.* D **65** 125027
[9] Quesne C and Tkachuk V M 2005 *J. Phys. A: Math. Gen.* **38** 1743
[10] Fityo T V, Vakarchuk I O and Tkachuk V M 2005 (*Preprint* quant-ph/0507117)
[11] Quesne C and Tkachuk V M 2003 *J. Phys. A: Math. Gen.* **36** 10373
[12] Quesne C and Tkachuk V M 2004 *J. Phys. A: Math. Gen.* **37** 10095
[13] Kempf A 1997 *J. Phys. A: Math. Gen.* **30** 2093
[14] Brau F 1999 *J. Phys. A: Math. Gen.* **32** 7691
[15] Benczik S, Chang L N, Minic D and Takeuchi T 2005 *Phys. Rev.* A **72** 012104
[16] Gordeyev A N and Chhajlany S C 1997 *J. Phys. A: Math. Gen.* **30** 6893
[17] Landau L D and Lifshits E M 1977 *Quantum Mechanics: Non-Relativistic Theory* (New York: Pergamon Press)